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Divergence Theorem

Idea: Generalize Green's Theorem again: This time the version

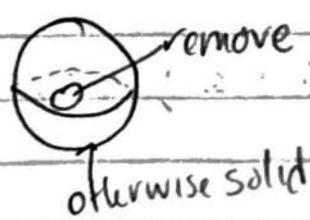
$$\int_{\partial D} \vec{F} \cdot \vec{n} \, ds = \iint_D \operatorname{div}(\vec{F}) \, dA$$

Proposition: (Divergence Theorem) – Suppose R is a solid region in $\mathbb{R}^3$ and $\vec{F}$ is a vector field which has cts. partials on R. If R is a simple region, then

$$\iint_{\partial R} \vec{F} \cdot d\vec{s} = \iiint_R \operatorname{div}(\vec{F}) \, dV$$

NB: A simple region in $\mathbb{R}^3$ is a solid w/ 1 boundary component (i.e. ∂R is a single surface) which is piecewise smooth

Non-ex



Not a simple region b/c it has 2 boundary components

Ex

solid disk

Ex: Compute the Flux of the v.f. $\vec{F} = \langle x, y, z \rangle$ across $x^2+y^2+z^2=1$

sol: Apply the divergence Theorem

$$\operatorname{div}(\vec{F}) = \nabla \cdot F = \frac{d}{dx}[x] + \frac{d}{dy}[y] + \frac{d}{dz}[z] = 3$$

Noting $S = \partial R$ for R the solid disk $x^2+y^2+z^2 \le 1$

$$\iint_S \vec{F} \cdot d\vec{s} = \iint_{\partial R} \vec{F} \cdot ds \underset{\text{div The.}}{=} \iiint_R \operatorname{div}(\vec{F})\, dV$$

$$= \iiint_R 3\, dV \quad 3\iiint_R 1\, dV = 3\operatorname{vol}(R) =$$

$$3\left(\tfrac{4}{3}\pi 1^3\right) = 4\pi$$

Verify w/ direct computation of surface integral

Sol 2:

$$S(\theta, \varphi) = \langle \sin(\varphi)\cos\theta, \sin(\varphi)\sin(\theta), \cos(\varphi) \rangle$$
on $(\theta, \varphi) = [0, 2\pi] \times (0, \pi]$

$\vec{S}_\theta = \langle -\sin(\varphi)\sin(\theta), \sin(\varphi)\cos(\theta), 0 \rangle$
$\vec{S}_\varphi = \langle \cos(\varphi)\cos(\theta), \cos(\varphi)\sin(\theta), -\sin(\varphi) \rangle$

$\therefore \vec{S}_\theta \times S_\varphi = \det \begin{vmatrix} i & j & k \\ -\sin\varphi\sin\theta & \sin\varphi\cos\theta & 0 \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \end{vmatrix}$

$= \langle -\sin^2\varphi\cos\theta, -(\sin^2\varphi\sin\theta), -\sin\varphi\cos\varphi\sin^2\theta - \sin\varphi\cos\varphi\cos^2\theta \rangle$

$= -\sin\varphi \langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi \rangle$

at $\theta = 0$ $\varphi = \frac{\pi}{2}$ we get $-\langle 1, 0, 0 \rangle$
pointing inward, so we use $-\vec{S}_\theta \times \vec{S}_\varphi$ for correct orientation

$\therefore \vec{F}(\vec{S}(\theta, \varphi)) \cdot -(\vec{S}_\theta \times \vec{S}_\varphi) =$

$\langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi \rangle \cdot$
$-\sin\varphi \langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi \rangle$

$= \sin\varphi \langle \sin^2\varphi\cos^2\theta + \sin^2\varphi\sin^2\theta + \cos^2\varphi \rangle =$
$\sin\varphi$

$\therefore \iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{S}(\theta, \varphi)) \cdot -(\vec{S}_\theta \times \vec{S}_\varphi) \, dA$
$= \iint_D \sin\varphi \, dA = \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi} \sin\varphi \, d\varphi \, d\theta =$

$$\int_{\theta=0}^{2\pi}\left[-\cos\varphi\right]_0^{\pi} = \int_{\theta=0}^{2\pi}(1+1)\,d\theta = 2\int_0^{2\pi} d\theta =$$

$$2\cdot 2\pi = \boxed{4\pi}$$

Note: The two solutions gave the same answer so we verified the divergence theorem

<u>Ex</u>: Calculate the Flux of $\vec{F} = \langle xe^y, z-e^y, -xy \rangle$ across the ellipsoid $x^2+2y^2+3z^2=4$

<u>Sol</u>: Apply the divergence Theorem:

$$\iint_S \vec{F}\cdot d\vec{s} = \iiint_R \operatorname{div}\vec{F}\,dV$$

for R the solid ellipsoid $x^2+2y^2+3z^2 \leq 4$

but $\operatorname{div}(\vec{F}) = \nabla\cdot\vec{F} = e^y - e^y + 0 = 0$

$\therefore \iint_S \vec{F}\cdot d\vec{s} = \iiint 0\,dV = 0$

<u>Note</u>: we could verify this one directly via parameterization of S... The parameterization is indicated by...

$x^2+2y^2+3z^2 = 4$ iff

$$\frac{x^2}{6} + \frac{y^2}{3} + \frac{z^2}{2} = \frac{2}{3}$$

iff $\left(\frac{x}{\sqrt{6}}\right)^2 + \left(\frac{y}{\sqrt{3}}\right)^2 + \left(\frac{z}{\sqrt{2}}\right)^2 = \left(\sqrt{\frac{2}{3}}\right)^2$

★ we should parameterize the solid disk using a version of spherical coordinates

$$\begin{cases} x = \sqrt{6}\rho \sin\varphi \cos\theta \\ y = \sqrt{3}\,\rho \sin\varphi \cos\theta \\ z = \sqrt{2}\,\rho \cos\varphi \end{cases}$$

yields $\rho^2 = \frac{2}{3}$ $\left(\text{check } \left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right| = 6\rho^2 \sin(\varphi)\right)$

Ex: Compute Flux of $\vec{F} = \langle 3x, xy, 2xz \rangle$ across the boundary $\partial[0,1]^3$

NB: Parameterizing this surface would require 6 different pieces...
- But the divergence Theorem, might not have to

Sol: Applying the divergence theorem:

$$\iiint_{\partial[0,1]^3} \vec{F}\cdot d\vec{S} = \iiint_{[0,1]^3} \operatorname{div}(\vec{F})\,dV$$

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{\partial}{\partial x}[3x] + \frac{\partial}{\partial y}[xy] + \frac{\partial}{\partial z}[2xz] = 3 + x + 2x = 3x + 3$$

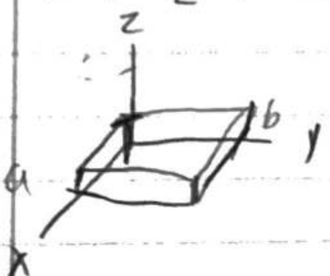
$$\therefore \iint_{\partial[0,1]^3} \vec{F}\cdot d\vec{s} = \int_{z=0}^{1}\int_{y=0}\int_{x=0}^{1} 3x+3\,dx\,dy\,dz$$

$$= \int_{z=0}^{1} \int_{y=0}^{1} \left[3x + \frac{3}{2}x^2\right]_0^1 \, dy\, dz$$

$$= \int_{z=0}^{1} \int_{y=0}^{1} 3 + \frac{3}{2} - 0 \, dy\, dz = 3\left(\frac{3}{2}\right) \text{Area}[0,1]^2$$

$$= \frac{9}{2}(1-0)(1-0) = \frac{9}{2}$$

Ex: Calculate the flux of $\vec{F} = \langle x^2yz, xy^2z, xyz^2 \rangle$ across the boundary of the rectangular box $R: [0,a] \times [0,b] \times [0,c]$ for $a,b,c > 0$



sol: lets apply the divergence Theorem

$$\iint_{\partial R} \vec{F} \cdot d\vec{s} = \iiint_R \operatorname{div}(\vec{F})\, dV$$

$$\operatorname{div} \vec{F} = 2xyz + 2xyz + 2xyz = 6xyz$$

$$\therefore \iint_{\partial R} \vec{F} \cdot d\vec{s} = \iiint_R 6xyz \, dV =$$

$$\int_{x=0}^{a} \int_{y=0}^{b} \int_{z=0}^{c} 6xyz \, dz\, dy\, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{b} 3\left[xyz^2\right]_0^c \, dy\, dx$$

$$\int_{x=0}^{a} \int_{y=0}^{b} 3(xyc^2 - 0) =$$

$$3c^2 \int_{x=0}^{a} \int_{y=0}^{b} xy \, dy\, dx = 3c^2 \int_{x=0}^{a} \frac{1}{2} x \left[y^2\right]_0^b dx$$

$$= \frac{3}{2} c^2 \int_{x=0}^{a} x(b^2 - 0)\, dx = \frac{3}{2} b^2 c^2 \int_{x=0}^{a} x =$$

$$\frac{3}{2} b^2 c^2 \left[\frac{1}{2} x\right]_0^a = \boxed{\frac{3}{4} a^2 b^2 c^2}$$